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Non-singular, cosmological solutions for the coupled Dirac–Einstein equations

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Abstract. The coupled Dirac–Einstein equations for an open Robertson–Walker universe admit a discrete spectrum of non-singular recollapsing solutions with associated finite lifetimes. This spectrum can be classified by topological quantum numbers, and the lifetime is roughly proportional to these numbers. The rather complicated structure of the spectrum is due to the dynamics of the relative phase angle with respect to the positive and negative energy components of Dirac's spinor field. The spectrum is characterized by a band structure: the phase angle remains in each allowed band for a relatively long time but then suddenly jumps to another one.

1. Introduction: Dirac–Einstein universe

As an alternative to the well known inflation scenarios [1] for the early universe [2], the Dirac–Einstein model [3–8] was recently established which is based upon the (minimal) coupling of Dirac's equation

$$i\hbar\gamma^{\mu}\mathcal{D}_{\mu}\psi = Mc\psi \tag{1}$$

to Einstein's equation

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi\frac{{L_p}^2}{\hbar c}\left(T_{\mu\nu} + \lambda_{*}g_{\mu\nu}\right) \tag{2}$$

for a homogeneous, isotropic universe

$$ds^2 = d\tau^2 - \mathcal{R}^2\{dr^2 + \sinh^2 r\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right)\} \tag{3}$$

(FRW universe [9]). The interesting point here is that the Dirac–Einstein model is in some sense 'complementary' to the inflation model, because it favours an *open, oscillating* universe with *negative* cosmological constant ($\lambda_{*} < 0$) [5]. In contrast to this, the idea of inflation is based upon a *flat, exponentially growing* universe implying a *positive* cosmological constant. In the present paper, we restrict ourselves to the purely mathematical side of the problem, which is interesting in itself, and we will obtain new solutions of the Dirac–Einstein equations (1)–(3), the properties of which are then investigated in some detail. We find that there is a discrete spectrum of solutions with finite lifetime in which all physical quantities (e.g. pressure and energy density) remain non-singular even in the limit of vanishing scale factor ($\mathcal{R} \to 0$). Thus the universe is *created ex nihilo* and subsequently *annihilates in nihilo* with the typical lifetime being quantized in units of $(Mc/\hbar)^{-1}$. This is just the time needed for light to cross the Compton wavelength of the Dirac particle.

To begin with the 'ground state' of the discrete spectrum, note that the energy–momentum density $T_{\mu\nu}$ of the matter for a FRW universe must have the following specific form ('cosmological principle' [9]):

$$T_{\mu\nu} = \mathcal{M} b_\mu b_\nu - \mathcal{P}\mathcal{B}_{\mu\nu} \tag{4}$$

where the derivative of the Hubble flow vector b_μ is proportional to its orthogonal projector $\mathcal{B}_{\mu\nu}$, according to the high symmetry of such a universe:

$$\nabla_\mu b_\nu = H\mathcal{B}_{\mu\nu} \qquad H = \frac{\dot{\mathcal{R}}}{\mathcal{R}}. \tag{5}$$

Now it has been shown recently [3, 6] that such a tensor $T_{\mu\nu}$ with homogeneous energy density $\mathcal{M}$ and pressure $\mathcal{P}$ can be generated by a Dirac spinor field ψ only in an *open* universe. The corresponding result is [3, 8]

$$\mathcal{M} = 3\hbar c\rho\left(\frac{\cos\chi}{2\mathcal{R}} + \frac{Mc}{3\hbar}\right) \tag{6}$$

$$\mathcal{P} = \hbar c\rho\frac{\cos\chi}{2\mathcal{R}}. \tag{7}$$

Here, both thermodynamic state functions $\mathcal{M}$ and $\mathcal{P}$ have been expressed in terms of the scale factor $\mathcal{R}$, the scalar density $\rho = \bar{\psi}\psi$ of the spinor field ψ and the relative phase angle χ between the positive and negative energy components [6]. Because of the universe's homogeneity, all dynamical variables $\{\mathcal{M}, \mathcal{P}, \mathcal{R}, \rho, \chi\}$ are functions exclusively of cosmic time τ, where energy conservation implies

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\mathcal{M}\mathcal{R}^3\right) = -\mathcal{P}\frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{R}^3. \tag{8}$$

For an equation of state of the form

$$\mathcal{P} = \beta\mathcal{M} \tag{9}$$

the solution to (8) for constant coefficient β is

$$\mathcal{M} = \mathcal{M}_*\left(\frac{\mathcal{R}_*}{\mathcal{R}}\right)^{3(1+\beta)} \tag{10}$$

which yields

$$\mathcal{M} = \mathcal{M}_*\left(\frac{\mathcal{R}_*}{\mathcal{R}}\right)^{3} \tag{11}$$

for a matter-dominated universe ($\beta = 0$) and

$$\mathcal{M} = \mathcal{M}_*\left(\frac{\mathcal{R}_*}{\mathcal{R}}\right)^{4} \tag{12}$$

for a radiation-dominated universe ($\beta = \frac{1}{3}$). However, for the exotic value $\beta = -1$ equation (10) leads to a time-independent energy density ($\mathcal{M}_*$) and the energy–momentum density $T_{\mu\nu}$ (4) acquires the form of a cosmological constant:

$${}^{(*)}T_{\mu\nu} = \mathcal{M}_*\, g_{\mu\nu}\,. \tag{13}$$

Clearly, the thermodynamic coefficient β will in general not be a constant for the present Dirac–Einstein model. Nevertheless, we assume $\beta = -1$ as an ansatz for obtaining the desired *ground state*. From equations (6) and (7), this ansatz readily yields

$$\frac{\cos\chi}{\mathcal{R}} = -\frac{m}{2} \tag{14}$$

where

$$m^{-1} \equiv \left(\frac{Mc}{\hbar}\right)^{-1} \tag{15}$$

is the Compton wavelength. On the other hand, Dirac's equation (1) for the wavefunction ψ may be transcribed into equations of motion for the phase angle χ and scalar density ρ, [3, 8]:

$$\frac{\mathrm{d}\chi}{\mathrm{d}t} \equiv \dot{\chi} = 3\frac{\cos\chi}{r} + 2 \tag{16}$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} \equiv \dot{\rho} = 3\frac{H}{m}\rho + 3\rho\frac{\sin\chi}{r} \qquad (t := m\tau,\ r := m\mathcal{R}). \tag{17}$$

The obvious *non-singular* solution satisfying constraint (14) is

$$\chi(t) = \frac{\pi}{2} + \frac{t}{2}. \tag{18}$$

Thus, from (14) the (rescaled) scale factor is

$$r(t) = 2\sin\frac{t}{2} \tag{19}$$

and the lifetime ($t_L = m\tau_L$) of the 'ground-state' universe is ${}^{(0)}t_L = 2\pi$. Unfortunately, since the corresponding scalar density $\rho = \bar{\psi}\psi(\equiv \mathcal{R}^{-3}\mu)$ must be time independent, it cannot be determined in this special case from its equation of motion (17). Therefore, we have to resort to the Einstein equations (2), which in terms of the rescaled quantities $t = m\tau$, $r = m\mathcal{R}$, $\mu = \rho\mathcal{R}^3$, $q = mL_p$, and $k_* = \lambda_* m^{-4}(\hbar c)^{-1}$ become

$$\ddot{r} = -\frac{4\pi}{3}q^2\frac{\mu}{r^2}\left(1 + 3\frac{\cos\chi}{r}\right) + \frac{8\pi}{3}q^2 k_* r \tag{20}$$

$$\dot{r}^2 = 1 + 8\pi q^2\frac{\mu}{r}\left(\frac{\cos\chi}{2r} + \frac{1}{3}\right) + \frac{8\pi}{3}q^2 k_* r^2. \tag{21}$$

For the 'ground-state' solutions, (18) and (19) yield

$$\mu(t) = -\frac{3}{\pi q^2}\left(1 + \frac{32\pi}{3}q^2 k_*\right)\sin^3\frac{t}{2} \tag{22}$$

and so the constant value of the scalar density ρ is

$$\rho = -\frac{3m^3}{8\pi q^2}\left(1 + \frac{32\pi}{3}q^2 k_*\right). \tag{23}$$

Since ρ is required to be positive, the cosmological constant must satisfy

$$k_* < -\frac{3}{32\pi q^2}. \tag{24}$$

This result exemplifies the non-singularity of the energy density $\mathcal{M}$ (6) and pressure $\mathcal{P}$ (7) of the solutions to the coupled Dirac–Einstein equations (1) and (2), as opposed to the analogous singular ($\mathcal{R} \to 0$) results, equations (11) and (12), for the standard cosmological model. Clearly, the present result is a sufficient motivation to look for further non-singular solutions, which behave near the universe's birth ($t \to 0$) as follows [3]:

$$\chi(t) = \frac{\pi}{2} + \frac{t}{2} + \cdots \tag{25}$$

$$\mu(t) = \mu_c t^3 + \cdots \tag{26}$$

$$r(t) = t + \frac{\pi}{9}q^2(\mu_c + 4k_*)t^3 + \cdots. \tag{27}$$

Similar behaviour applies near its death ($t \to t_L$). Observe here that the ground-state solution (18), (19) and (22) fits into this scheme provided the integration constant μ_c adopts its ground-state value

$$^{(0)}\mu_c = -\frac{3}{8\pi q^2}\left(1 + \frac{32\pi}{3} q^2 k_*\right). \tag{28}$$

Thus, our problem is to look for the 'higher excited states' with values $^{(n)}\mu_c$ of the 'initial particle number' μ_c such that the energy density $\mathcal{M}$ (6) and pressure $\mathcal{P}$ (7) both remain finite for the whole lifetime $0 \leqslant t \leqslant^{(n)} t_L$. Obviously, the *non-singularity conditions*

$$\chi(0) = \frac{\pi}{2} \bmod 2\pi \qquad \chi(t_L) = \frac{3\pi}{2} \bmod 2\pi \tag{29}$$

act as a kind of quantization condition which selects the discrete values $^{(n)}\mu_c$ from the continuous range $0 \leqslant \mu_c < \infty$. The initial particle number μ_c may be considered as a 'coordinate' parametrizing the one-dimensional manifold of solutions according to (25)–(27). Consequently, the non-singular solutions constitute a discrete subset of this manifold.

In this paper we investigate the excited states of 'low order', but in order to do so we must first define the notion of 'order' for the expected discrete spectrum of solutions. Intuitively, one would like to count the discrete subset $^{(n)}\mu_c$ simply by $n = 0, 1, 2, 3, \ldots$. However, as we shall see, one needs to introduce more than a single 'quantum number' n.

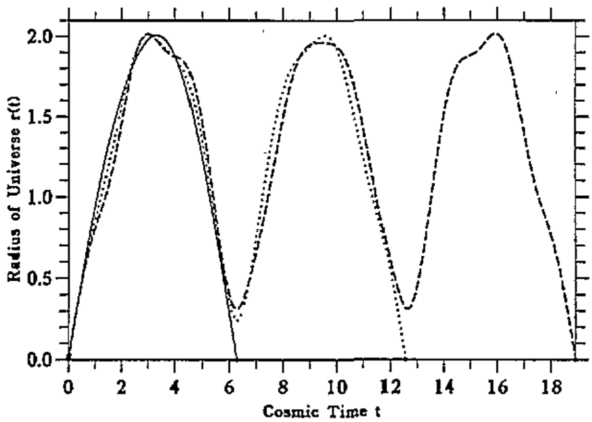

Figure 1. Ground-state and excitations. The non-singular excitations of the ground state (19) (full curve) exhibit some regular features, e.g. the number of bounces (0, 1, 2) or the lifetimes $t_L (= 2\pi, \approx 4\pi, \approx 6\pi)$, indicating the existence of some classification scheme. The bold dots symbolize the universe's creation and annihilation ($r = 0$). Parameters used in this figure are shown in table 1.

Table 1. Parameters used in figure 1.

	Full	Dotted	Broken
μ_c	1.0	1.0	1.0
q^2	0.298 41...	0.408 632...	0.247 545...
k_*	−0.4	−0.584 222 3...	−0.964 398 2...

2. Topological classes

Figure 1 shows three non-singular (numerical) solutions to the coupled Dirac–Einstein system (16), (17), (20), (21). The corresponding lifetimes are exactly 2π for the ground state (19) and approximately 4π and 6π for the excited states. By analogy with well known problems in elementary quantum mechanics (e.g. the square well potential or harmonic oscillator), one might anticipate that the present Dirac–Einstein solutions could be classified by the number of bounces occurring during their lifetime. The lifetime apparently correlates strongly with the bounce number. With this assumption, figure 1 would exhibit the three lowest-order solutions $n = 0, 1, 2$. However, can we be sure that continuously varying the parameters q and k_* in the Dirac–Einstein equations does not change the number of bounces for a non-singular solution satisfying (29)? Evidently, we need a *topological invariant* such that any solution with finite lifetime remains in its 'topological class' during continuous variation of the parameters q, k_*, μ_c.

For obtaining the appropriate topological criterion, it is better to look at the χ/t diagram (figure 2). In the ground state, the phase angle χ has the value π just once, in the 'first excited state' three times, in the 'nth excited state' it does so $2n+1$ times. Is the number of π crossings a topological invariant and thus suited for establishing a classification scheme? Fortunately yes, because in continuously deforming a solution with $(2n + 1)$ π-passages into one with $(2n - 1)$ π-passages, one must encounter a situation where $\chi = \pi$ and $\dot{\chi} = 0$ simultaneously. But this is forbidden by equations (16) and (21), provided

$$q^2 k_* < -\frac{1}{6\pi}. \tag{30}$$

In this case, the $\pi(\bmod 2\pi)$-passage number n_Π $(= 1, 3, 5, \ldots)$ is a good quantum number and may be used as a classification criterion. (During our numerical integrations, we never observed a breakdown of this classification, even when condition (30) did not hold.) One can easily show that n_Π is the number of times the particle number $\mu(t) := \rho\mathcal{R}^3$ attains a maximum ($\dot{\mu} = 0$). Unfortunately, the number of maxima turns out to be insufficient to establish a *unique* classification of the discrete spectrum and there is need for a further quantum number.

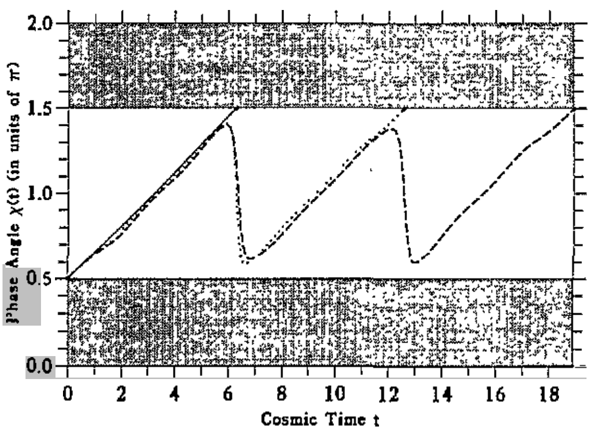

Figure 2. Passage number n_Π. The number n_Π of times when χ has the value $\pi(\bmod 2\pi)$ is an invariant under continuous variation of q, k_*, μ_c along $\tilde{R}_{\Pi,\Delta}$. For the solutions of figure 1, $n_\Pi = 1, 3, 5$. The ground state (full curve) has $n_\Pi = 1$. The initial and final values for the phase χ are kept fixed through the non-singularity condition (29).

3. Phase jumping

Evidently, the common feature of the solutions shown in figures 1 and 2 is the restriction of the variation of the phase angle χ to the domain $\frac{\pi}{2} \leqslant \chi \leqslant \frac{3\pi}{2}$. However, as indicated by figure 3, there exist solutions for which the angle χ leaves this domain and jumps in a relatively short time first to the domain $\frac{5\pi}{2} \leqslant \chi \leqslant \frac{7\pi}{2}$, and then to the domain $\frac{9\pi}{2} \leqslant \chi \leqslant \frac{11\pi}{2}$. It is a striking fact that the phase can stay for long time in the 'allowed bands' $(\frac{2n+1}{2}\pi \leqslant \chi \leqslant \frac{2n+3}{2}\pi)$, whereas it crosses the intermediate bands $(\frac{2n+3}{2}\pi \leqslant \chi \leqslant \frac{2n+5}{2}\pi)$ very rapidly. As the corresponding r/t diagram (figure 4) shows, such 'phase jumps' occur whenever the scale factor r tends to zero without the phase angle

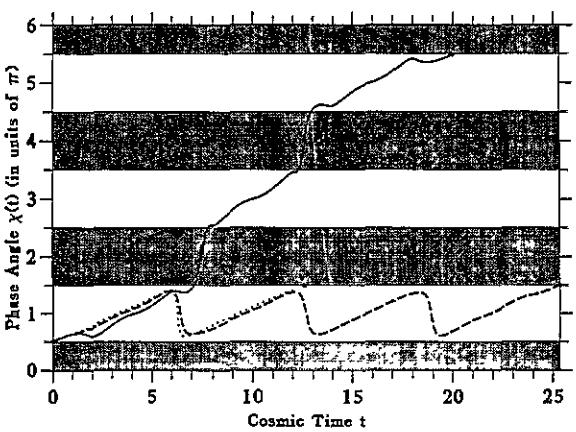

Figure 3. Phase jumping. Transitions of the angle χ to a neighbouring allowed band are possible and generate different solutions with the same number n_Π (full and dotted curves: $n_\Pi = 3$). But with respect to the quantum configuration (n_Π, n_Δ) all solutions are discernible; full curve: $(n_\Pi = 3, n_\Delta = 5)$, dotted curve: $(n_\Pi = 3, n_\Delta = 1)$, broken curve: $(n_\Pi = 7, n_\Delta = 1)$. The universe's lifetime roughly correlates with the sum $n_\Pi + n_\Delta$.

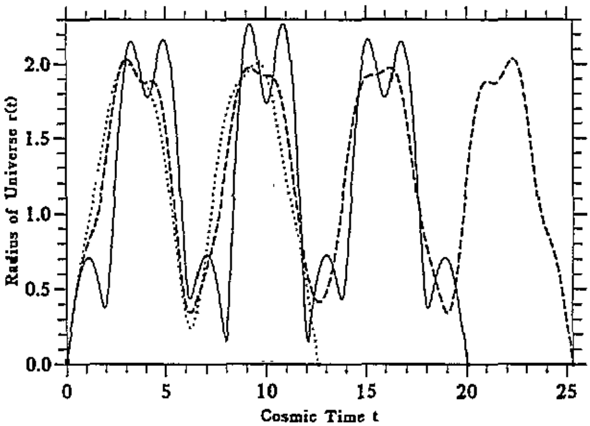

Figure 4. Second quantum number n_Δ. The two solutions of figure 3 with the same topological number $n_\Pi = 3$ (full and dotted curve) are different, e.g. the lifetimes are $t_L \approx 4\pi$ (dotted) and $t_L \approx 6\pi$ (full). This difference is expressed in the second quantum number n_Δ (full: $n_\Delta = 5$; dotted $n_\Delta = 1$).

χ satisfying the non-singularity condition (29). As a consequence, the phase jumps over to the neighbouring *allowed* band and tries to produce a non-singular collapse in that domain. The reason for this peculiar behaviour is readily seen from the equation of motion for χ (equation (16)): if a collapse occurs ($r \to 0$) with the phase χ not satisfying condition (29), but lying in an intermediate band ($\cos\chi > 0$), then equation (16) predicts a large angular velocity ($\dot{\chi} \gg 1$) resulting in a quick traversal of the intermediate band. Evidently the quicker the jump, the harder the bounce (see figure 4).

A consequence of phase jumping is that the topological number n_Π introduced above is too coarse, i.e. we encounter different solutions with the same number n_Π (cf figures 3 and 4). Therefore, in order to refine our classification, we have to introduce a further topological number. To this end, we exploit the non-singularity condition (29) and define a new quantum number

$$n_\Delta := \frac{\chi(t_L) - \chi(0)}{\pi}. \qquad (31)$$

The solutions of figures 3 and 4 are now distinguished by their quantum numbers (n_Π, n_Δ) (figure 3).

4. Asymmetric solutions

So far we have only considered time-symmetric solutions, i.e. there always existed an instant t_* such that $r(t_* - T) = r(t_* + T)$ with $t_* = t_L/2$ and $0 \leqslant T \leqslant t_L/2$. (This also holds for the particle number $\mu(t)$.) The question arises of whether the coupled Dirac–Einstein system also admits *asymmetric* solutions?

In order to answer this question, it is convenient to consider the two-dimensional parameter space $\tilde{P} = \{\tilde{\mu}, \tilde{\eta}\}$ with $\tilde{\mu} := |k_*|^{-1}\mu_c$, $\tilde{\eta} := \frac{8\pi}{3}q^2|k_*|$. The set of all points $\tilde{p} = (\tilde{\mu}, \tilde{\eta}) \in \tilde{P}$ in this 2-space for which a non-singular solution $(\tilde{r})$ exists forms a subset $\tilde{R} = \{\tilde{r}\}$ which is the union of the corresponding subclasses $\tilde{R}_{\Pi,\Delta}$ containing all the $(\tilde{r}_{\Pi,\Delta})$

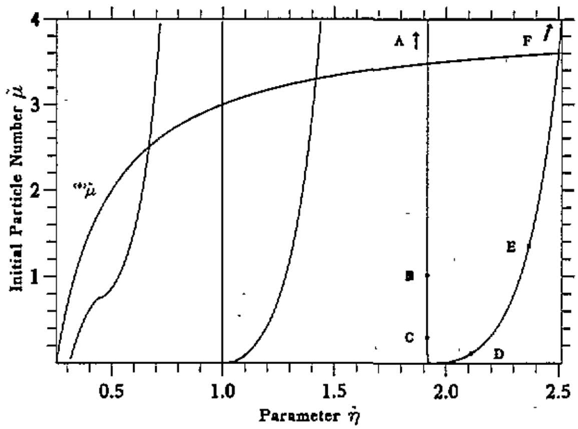

Figure 5. The non-singular class $\tilde{R}_{1,1}$. The set $\tilde{R}_{1,1}$ is connected in $\tilde{P}$ and admits a continuous deformation of any member $\tilde{r}_{1,1} \in \tilde{R}_{1,1}$ into the ground state (19), whose parameter curve ${}^{(0)}\tilde{\mu} = {}^{(0)}\tilde{\mu}(\tilde{\eta})$ is given by equation (32). The choice of the representative points for the route A, B, C, D, E, F is: $\tilde{r}_A = (1.916525\ldots, 10.0)$, $\tilde{r}_B = (1.916483\ldots, 1.0152)$, $\tilde{r}_C = (1.916227\ldots, 0.3)$, $\tilde{r}_D = (2.111, 0.1100985\ldots)$, $\tilde{r}_E = (2.3675, 1.347856\ldots)$, $\tilde{r}_F = (2.63793\ldots, 10.0)$.

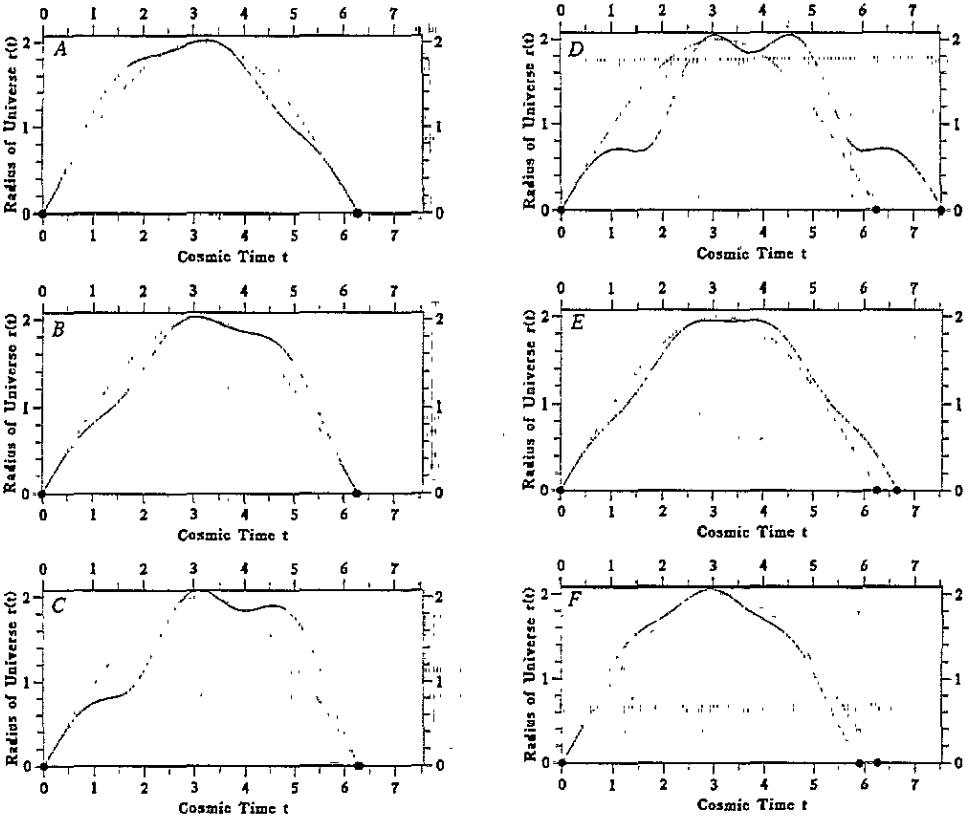

Figure 6. Asymmetric solutions $r(t)$. The left branches $A, B, C, \ldots$ of $\tilde{R}_{\mathrm{I,I}}$ contain the asymmetric solutions, with the exception of the intersection of such a branch with the ground state curve (32). The right branches $D, E, F \ldots$ consist of strictly symmetric solutions. Note that the lifetime of the *asymmetric* solutions is not changed.

solutions: $\tilde{R} = \cup_{\Pi,\Delta} \tilde{R}_{\Pi,\Delta}$. For the discussion of the asymmetry problem, we restrict ourselves to the simplest class ($n_\Pi = 1, n_\Delta = 1$) which contains the ground state (equation (19)). The corresponding parameter manifold $\tilde{R}_{\mathrm{I,I}}$ is exhibited in figure 5.

Evidently, $\tilde{R}_{\mathrm{I,I}}$ is a connected one-dimensional subset of the parameter space $\tilde{P}$. Consequently, we can join any solution $\tilde{r}_{\mathrm{I,I}} \in \tilde{R}_{\mathrm{I,I}}$ to the symmetric ground state (19) by continuous variation of the parameters $(\tilde{\mu}, \tilde{\eta})$, but this does not imply that $\tilde{R}_{\mathrm{I,I}}$ only contains symmetric solutions. A simple counterexample is characterized by taking the route $A \to B \to C \to D \to E \to F$ in figure 5. The corresponding solutions $r_A(t) \ldots r_F(t)$ for the scale factor r are shown in figure 6, from which the result can easily be read off: for the left branch $C, B, A, \ldots$ the solutions are *asymmetric*, whereas the right branch $D, E, F, \ldots$ contains the *symmetric* solutions. Clearly, whenever some branch of $\tilde{R}_{\mathrm{I,I}}$ intersects with the ground-state branch (28):

$$^{(0)}\tilde{\mu} = \left(4 - \frac{1}{\tilde{\eta}}\right) \tag{32}$$

then a symmetric solution is encountered. Observe also the different kind of asymmetry for $\tilde{\mu} >^{(0)} \tilde{\mu}$ and $\tilde{\mu} <^{(0)} \tilde{\mu}$ in the left branch $\ldots A, B, C$.

We have restricted here our discussion to the lowest order class $\tilde{R}_{\mathrm{I,I}}$ because the higher classes $\tilde{R}_{\Pi,\Delta}$ with $n_\Pi > 1$, $n_\Delta > 1$ are no longer connected and exhibit a very complicated *fractal* structure (to be investigated in a separate paper).

5. Abundance ratio of the quantum numbers n_Π

It should be clear from the preceding discussion that there is no restriction on the second quantum number n_Δ, so we can envisage non-singular solutions with a large value for n_Δ. This means that the phase angle χ traverses a large number of allowed bands ($n_\Delta \gg 1$). On the other hand, the first number n_Π can never be smaller than the second number n_Δ, so we must also have a large value for n_Π (as an example of such a solution see figure 7). Since n_Π may be considerably larger than n_Δ, there arises the question of how the passage numbers n_Π are distributed over the allowed bands.

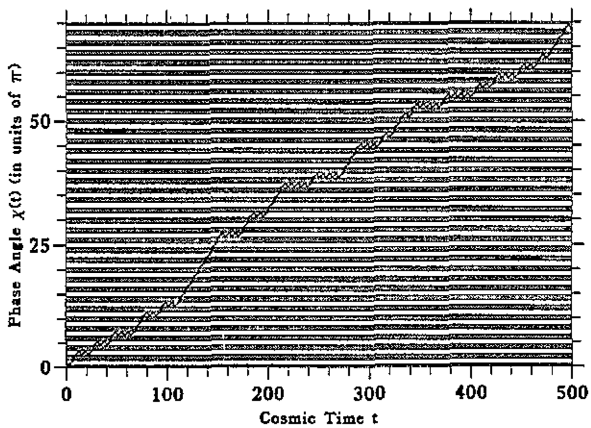

Figure 7. Solutions with large quantum numbers. For long-lived universes (here $t_L \approx 2063$), the quantum numbers n_Π, n_Δ are arbitrarily large. The solution has $n_\Pi = 513$, $n_\Delta = 297$, where the maximal value of n_Π per allowed band is $(n_\Pi)_{max} = 9$. The minimal value is always $(n_\Pi)_{min} = 1$.

A numerical investigation of this question reveals, that for any solution there is a preferred abundance of n_Π per band, where the preferred value may differ from one solution to another. Two typical situations are encountered in figures 8 and 9. In the case of figure 8, the n_Π exhibits two preferred values per band, namely 1 and 13; in the case of figure 9 one finds the single preferred value $n_\Pi = 1$ and the relative abundance of higher n_Π decreases monotonically.

Whether the band number n_Π has a regular or chaotic distribution is strongly related to the problem of matter production in the early universe. It has been shown [5] that matter *production* through Dirac's spinor field can occur only during the short time interval when the scale factor of the universe is close to minimal, i.e. during the bounce. However, the Dirac–Einstein system predicts that matter *annihilation* is equally possible during that short time interval. The reason is that (8) requires the pressure $\mathcal{P}$ (7) and the expansion rate H to have different signs in order that matter be produced. Since the expansion rate is always positive in a forbidden band, matter can be produced only in the *allowed* bands, preferably in the vicinity of $\chi = \pi$ during a bounce. However, when the time behaviour during the bounce is completely *symmetric* with respect to $\chi = \pi$, the matter produced shortly before the bounce ($\chi \leqslant \pi$) is annihilated again shortly after the bounce ($\chi \geqslant \pi$) [4]. Consequently,

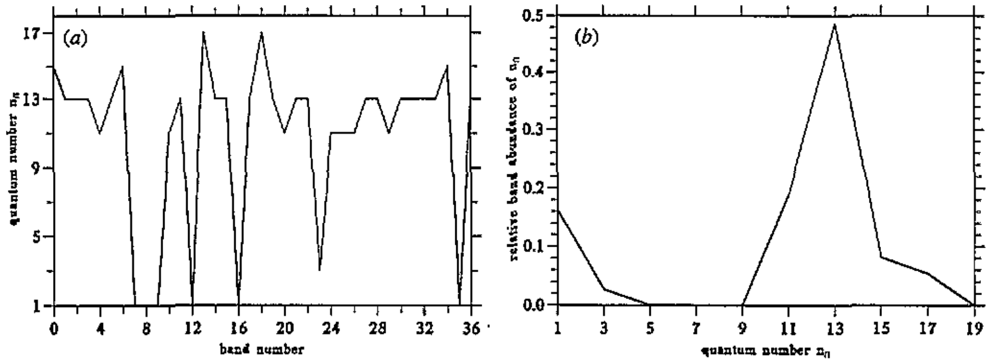

Figure 8. Relative abundance of the band number n_π. The number of π passages (mod 2π) in any band fluctuates irregularly (a) but the statistical analysis (b) reveals the occurrence of two preferred values, $n_\pi = 1$ and $n_\pi = 13$.

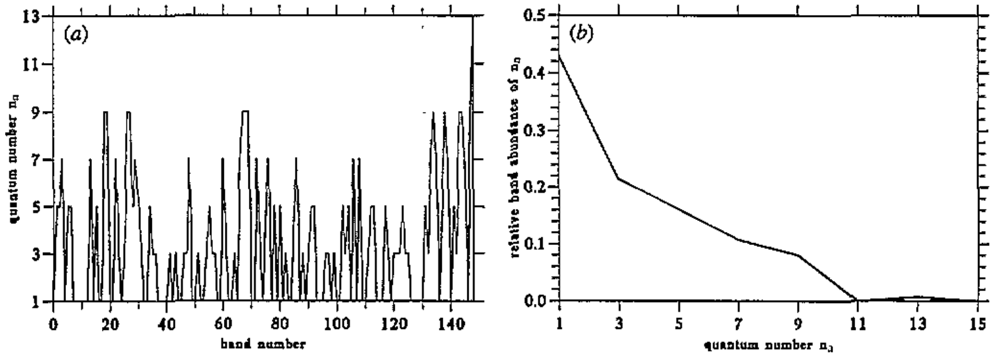

Figure 9. Statistics of the band number n_π. The quantum number n_π per band may obey quite different types of statistics for different solutions. In contrast to the solution of figure 8, there is a single preferred value for n_π, namely $n_\pi = 1$. Higher values of n_π are unlikely.

when the bounces occur chaotically, one expects matter production to be of equal strength to matter annihilation for a large number of bounces. However, if one could find some statistical regularity for the bounces, then one could infer that the matter content of the universe built up through continuous bouncing; the reason is that the matter-*producing* bounces had some statistical dominance over the matter-*annihilating* bounces. Note that the question of matter production is closely related to the maximal size of the universe: the greater the particle number μ the greater the maximal scale factor r, cf Einstein's equation (21). A more thorough analysis of the chaotic properties of the Dirac–Einstein system (16), (17); (20), (21) would now appear desirable.

Acknowledgments

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